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LETTER TO THE EDITOR

# Randomness in vertex models and directed walks

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**Abstract.** We consider a $d$-dimensional random five vertex (modified KDP) model where the vertex energies are site dependent, uncorrelated random numbers ($>0$). This model maps onto many directed walks in a random environment. We show that the upper critical dimension of the random vertex model is 2. We obtain a bound $\nu \geq 3/(d+3)$ for the size exponent of a directed walk in a random medium. The breakdown of hyperscaling in the vertex model is connected to the anomalous growth of the free energy with an exponent consistent with the corresponding one ($\chi = 2 - 1/\nu$) for a single directed walk.

The directed random walk (DRW), over the last few years, has turned out to be the unifying framework for widely different topics like vertex models [1, 2], commensurate-incommensurate transitions (CIT) [3], biomembrane transitions [4], flux-lattice melting [5, 6], surface growth [7, 8], critical behaviour of disordered systems, spin-glasses [9, 10] etc. Despite the current controversies, there is a consensus that the directed walk in a random environment (to be called Ran-D-walk) is the simplest model showing the features and complexities of critical behaviour of random systems. In this letter we explore the connection between Ran-D-walks and a random† ferroelectric five vertex or the modified KDP model (see figure 1) on a $d$-dimensional diamond type lattice [1, 2]. Similar two-dimensional models have been studied in the past [3, 11] in connection with CIT, though a major impetus for such models, Ran-D-walk in particular, is the potential application in flux-lattice melting (FLM) in impure (i.e. realistic) high $T_c$ superconductors. The distinctive feature of the vertex model *vis-à-vis* FLM is that in the former, temperature controls both the density of walkers (conjugate to $T - T_c$, for FLM conjugate to $H - H_{c_1}$) and the fluctuation.

For any $d$, the system remains frozen in the ground state for $T < T_c$, and the directed walk type excitations (figure 1) determine the thermodynamic behaviour of the high temperature phase. This line density $\rho$ ($\equiv n/V_\perp$, $n$ being the number of walkers, $V_\perp$

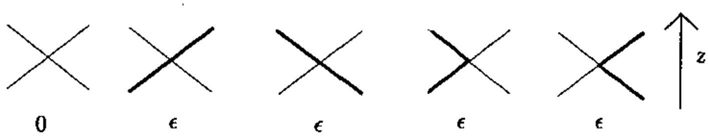


**Figure 1.** Five vertices for the five vertex model. It costs an energy $\varepsilon$ to create a DRW (thicker line) at a site and each DRW traverses the whole lattice in the $z$ direction. $\varepsilon(>0)$ is taken to be a random number of finite variance.

† The detailed distribution of the randomness is not crucial, except probably the validity of the central limit theorem (see below).

the transverse volume) goes to zero as $T \to T_c+$. Our focus is only on the critical behaviour around this point. $T_c$ can be determined, like the pure case [1, 2], by a stability analysis of the ground state against the directed walk type excitations. The thermodynamics is obtained by minimizing the free energy (making use of the conservation of lines in the $z$-direction; see [2, 4] for details)

$$f(\rho) \sim -t\rho + s(\rho) \tag{1}$$

where $s(\rho)$ is the excess free energy due to the non-overlap constraint in the lattice model (figure 1). In an equivalent continuum formulation, this function $s(\rho)$ can be obtained from the $n$ chain Hamiltonian [5, 6]

$$\mathscr{H}_n = \frac{1}{2}\sum_{\alpha=1}^{n} \int dz \left[\left(\frac{\partial \boldsymbol{r}_\alpha}{\partial z}\right)^2 + V_D(\boldsymbol{r}_\alpha(z), z)\right] + v \sum_{\alpha<\beta} \int dz\, \delta^{d-1}(\boldsymbol{r}_\alpha(z) - \boldsymbol{r}_\beta(z)) \tag{2}$$

where $\boldsymbol{r}_\alpha(z)$ is the $d-1$ dimensional coordinate of point at a contour length $z$ of chain $\alpha$, and the $z$ integration is over the chain length that goes to infinity in the thermodynamic limit of the vertex model. The $v$ term is the mutual repulsion that leads to the fluctuation dominated critical behaviour in the pure system. The random potential $V_D$ is taken to be a random, uncorrelated function of $\boldsymbol{r}$ and $z$, with $\overline{V_D(\boldsymbol{r}, z) V_D(\boldsymbol{r}', z')} = \Delta\delta(\boldsymbol{r} - \boldsymbol{r}')\delta(z - z')$ (overbar denotes average with respect to the disorder distribution). The pure problem corresponds to $\Delta = 0$. The $v$ dependent excess free energy (after averaging over disorder) over and above the independent chain free energy gives $s(\rho)$ of (1) (see [2] for details).

We will not go into a detailed evaluation of $s(\rho)$ here, but rather use the scaling results of [11], concentrating on the exponents that are analogous to the pure case. These are (i) the 'incommensuration' exponent $\bar{\beta}$, $\rho \sim t^{\bar{\beta}}$ where $t \equiv T - T_c$, (ii) length scale exponents $\nu_\perp$ and $\nu_\parallel$: $\xi_\perp(\sim t^{-\nu_\perp})$, giving the average spacing between two walkers, and $\xi_\parallel(\sim t^{-\nu_\parallel})$ giving the average separation in the preferred $z$ direction between two collisions with neighbouring walkers, and (iii) the specific heat exponent $\alpha = 1 - \bar{\beta}$. In addition, we need two exponents for DRW, the size (or 'roughness') exponent $\nu$, $\langle R^2 \rangle \sim N^{2\nu}$ where $\boldsymbol{R}$ is the end-to-end distance in the transverse direction for a length $N$, and the free energy fluctuation exponent $\chi$ [7]. For the pure case, $\nu = \frac{1}{2}$, $\chi = 0$ are the random walk values, but for the random problem, the nature and value of $\nu$ are still under debate. The various estimates are: $\nu = 3/(d+3)$ (Flory type, see [11]) $\frac{2}{3}$ (superuniversal; see [11]), $(d+2)/2(d+1)$ [8], and a few others [6]. Of all these, $\nu(d=1) = \frac{3}{4}$ and $\nu(d=2) = \frac{2}{3}$ are considered to be exact [6, 8].

A scaling argument connects all the vertex model exponents, $\bar{\beta}$, $\nu_\perp$, $\nu_\parallel$ and $\alpha$ with the size exponent $\nu$ [2, 11]:

$$\bar{\beta} = \frac{(d-1)\nu}{2(1-\nu)} \qquad \nu_\perp = \nu\nu_\parallel = \frac{\nu}{2(1-\nu)} \qquad \bar{\beta} = 1 - \alpha. \tag{3}$$

Extensive RG analysis for the pure case ($\nu = \frac{1}{2}$) proved that all of these relations are *exact* [2]. Recently Natterman *et al* [6] showed that the relation for $\bar{\beta}$ is true as well for the random problem, corroborating, in turn, the relations for $\nu_\perp$ and $\nu_\parallel$, since the scaling analysis is built on these. Furthermore, the exponents for the pure case were shown in [1] to satisfy the anisotropic hyperscaling. However, extending that to the random problem was not justified. In this letter, we show a few simple but significant consequences of the scaling formulae, equation (3), and address the question of breakdown of hyperscaling, thereby correcting an error in the estimate of the upper critical dimension for the random problem in [1]. We also give the exact solution of the one-dimensional random vertex model to show that $\alpha = 1$ (i.e. $\tilde{\beta} = 0$) consistent with equation (3).

*Upper critical dimension.* The upper critical dimension is obtained by equating $\alpha = 0$ or $\tilde{\beta} = 1$†. This gives from (3) $d_c = (2/\nu) - 1$, as found by Natterman *et al* and the Kim-Kosterlitz formula for $\nu$ locates $d_c = 2$ [6]. (Incidentally, hyperscaling in [1] gave $d_c = (3/\nu) - 3$ which with superuniversality of $\nu$ gave $d_c = 1.5$. We now believe that this is not correct.)

*Hyperscaling and bound on* $\nu$. For isotropic disordered systems, Chayes *et al* proved that $d\nu \geqslant 2$, where $\nu$ is the appropriate length scale exponent [12]. Recently Schwartz proved this formula under quite general conditions [13]. This latter proof can very easily be extended to anisotropic systems to yield $(d-1)\nu_\perp + \nu_\parallel \geqslant 2$ and using (3) we get

$$\nu \geqslant \frac{3}{d+3}. \tag{4}$$

In other words the size exponent for a Ran-D-walk is bounded below by the Flory exponent (which is exact for $d = 1$). (A similar bound has also been obtained in a different way in [14].) That this bound is the Flory exponent is not accidental but is intimately connected to the neglect of the anomalous fluctuation in energy (see below). This anomalous growth of the free energy scale in a correlation volume leads to the breakdown of hyperscaling in random systems [15]. We now show that this is also the case for this random vertex model.

Taking a modified hyperscaling relation as

$$2 - \alpha = (d-1)\nu_\perp + \nu_\parallel + \Gamma \tag{5}$$

we find, from the exponents in (3) that $\Gamma = (1-2\nu)/2(1-\nu)$. The breakdown exponent $\Gamma$ vanishes if and only if $\nu = \frac{1}{2}$ (i.e. a pure system). For pure systems with $d > d_c = 3$, hyperscaling is expected to fail and it is a trivial exercise to check in this case that the dangerous irrelevant variable $v$ modifies the hyperscaling relation through a singular scaling function. In contrast, we see a failure even below the upper critical dimension for the random system ($\nu \neq \frac{1}{2}$).

To get to the origin of the breakdown, we note that a Ran-D-walk is isomorphic to a nonlinear Burgers equation or a surface growth problem [7, 11]. These latter problems have been studied using RG [7]. The main results, when translated back to the Ran-D-walk case, show that under a rescaling $\boldsymbol{r} \to b_\perp \boldsymbol{r}$, two scale factors are needed, one for the $z$-direction as $z \to b_\perp^{1/\nu} z$ (i.e. $b_\parallel = b_\perp^{1/\nu}$: this gives the factor of $\nu$ in $\nu_\perp = \nu\nu_\parallel$; see (3)) and the other for the free energy fluctuation which scales by a factor $b_\perp^{\chi}$ (height $h$ in [7]). For uncorrelated noise, the two exponents are related by $\chi = 2 - 1/\nu$. In recent numerical simulations on the *overlap* of two Ran-D-walks, Mézard also found this scaling of energy to be essential to explain the data. (Note that $\chi = 0$ for the pure case ($\nu = \frac{1}{2}$).) Hence, near the critical point, the free energy should behave like

$$f \sim b_\perp^{-(d-1)} b_\parallel^{-1} b_\perp^{\chi}. \tag{6a}$$

Choosing $b_\perp \sim t^{-\nu_\perp}$, $b_\parallel \sim t^{-\nu_\parallel}$ with $\nu_\perp = \nu\nu_\parallel$ as in (3) we obtain

$$2 - \alpha = (d-1)\nu_\perp + \nu_\parallel - \nu_\perp\chi. \tag{6b}$$

The breakdown exponent $-\nu_\perp\chi = (1-2\nu)/2(1-\nu)$ matches exactly with $\Gamma$ from (5).

† Note that $\alpha = 0$ is the mean field result. It follows from (3), whence $s(\rho) = v\rho^2$, if fluctuation in $\rho$ is ignored.

*One-dimensional model.* The general formula for $\bar{\beta}$ in (3) shows that $\alpha = 1$ for $d = 1$ for both pure [16] and random problem; the latter case is studied here.

Let us consider a one-dimensional five vertex model, with random site dependent energy $\varepsilon_i$ (with mean $\bar{\varepsilon}$ and finite variance). For a given realization $a$, it is easy to show [16] that for $N$ sites, the thermal average is

$$\langle E_a \rangle = \frac{E_a \exp(-\beta E_a + N \ln 2)}{1 + \exp(-\beta E_a + N \ln 2)} \qquad E_a = \sum \varepsilon_i. \tag{7}$$

Since the thermal energy depends only on the total energy $E_a$, the central limit theorem ensures that in the limit $N \to \infty$, the *quench averaged* energy is

$$\lim_{N\to\infty} \frac{\overline{\langle E \rangle}}{N} = \lim_{N\to\infty} \frac{\bar{\varepsilon} \exp[-N(\beta\bar{\varepsilon} - \ln 2)]}{1 + \exp[-N(\beta\bar{\varepsilon} - \ln 2)]}. \tag{8}$$

This indicates a first order phase transition at $T_c = \bar{\varepsilon}/k \ln 2$ and $f \sim (T - T_c)$ with $\alpha = 1$. Note that this is an example where randomness does not round a first order transition [17].

We summarize that the general 'hyperscaling type' inequality for random vertex models yields a bound on the geometric exponent $\nu$ of a Ran-D-walk, $\nu \geqslant 3/(d+3)$. The lower bound is the Flory exponent [11]. The random vertex model has an upper critical dimension 2. The breakdown of the hyperscaling relation for the random problem has been shown to be related to the anomalous growth of the free energy scale, the appropriate exponent being $\chi = 2 - 1/\nu$ [7, 9], which is zero for the pure case. This also indicates that the free energy fluctuation exponent for directed walks in random media is the same for a single walk as for a finite density case—in other words, no new exponent is needed for the finite density system. We also proved the existence of a first order transition in the random one-dimensional model, consistent with the general formula for the exponents in (3).

We end with two speculations. The anomalous growth of the free energy can lead to extremely slow dynamics [15] and it is tempting to surmise such a behaviour for the vertex model and the flux lattice melting problem. Another interesting conjecture is the possibility of a disorder dominated intermediatee phase for $d \geqslant 2$, where the second transition to a 'pure type' high temperature phase ($\nu = \frac{1}{2}$) could come about through depinning [18] of the Ran-D-walks. We would like to come back to these problems elsewhere.

## References

[1] Bhattacharjee S M 1991 *Europhys. Lett.* **15** 815
[2] Rajasekaran J J and Bhattacharjee S M 1991 *Phys. Rev.* A (in press); 1991 *J. Phys. A: Math. Gen.* **24** L371-5
[3] Kardar M 1987 *Nucl. Phys.* B **290** [FS20] 582 and references therein
[4] Izuyama T and Akutsu Y 1982 *J. Phys. Soc. Japan* **51** 50 and references therein
[5] Nelson D R and Seung H S 1989 *Phys. Rev.* B **39** 9153
Nelson D R 1988 *Phys. Rev. Lett.* **60** 1973
[6] Natterman T, Feigelman M and Lyuksyutov I 1991 *Z. Phys.* B **84** 353
[7] Kardar M, Parisi G and Zhang Y C 1986 *Phys. Rev. Lett.* **56** 889
[8] Kim J M and Kosterlitz J M 1989 *Phys. Rev. Lett.* **62** 2289
Kim J M, Moore M A and Bray A J 1991 *Phys. Rev.* A **44** 2345 and references therein

[9] Mézard M 1990 *J. Physique* **51** 1831
[10] Fisher D S and Huse D A 1991 *Phys. Rev.* B **43** 10728 and references therein
[11] Kardar M 1987 *J. Appl. Phys.* **61** 3601; 1990 *New Trends in Magnetism* ed M D Cutinho-Filho and S M Rezende (Singapore: World Scientific)
[12] Chayes J T, Chayes L, Fisher D S and Spencer T 1986 *Phys. Rev. Lett.* **57** 2999
Singh R R P and Fisher D S 1988 *Phys. Rev. Lett.* **60** 548
[13] Schwartz M 1991 *Europhys. Lett.* **15** 777
[14] Le Doussal P and Machta J 1991 *J. Stat. Phys.* **64** 541
[15] Villain J 1985 *J. Physique* **46** 1843
Bray A J and Moore M A 1985 *J. Phys. C: Solid State Phys.* **18** L927
Fisher D S 1986 *Phys. Rev. Lett.* **56** 416
[16] Nagle J F 1968 *Am. J. Phys.* **36** 1114
[17] Aizenman M and Wehr Jan 1989 *Phys. Rev. Lett.* **62** 2503
[18] Derrida B and Golinelli O 1990 *Phys. Rev.* A **41** 4160